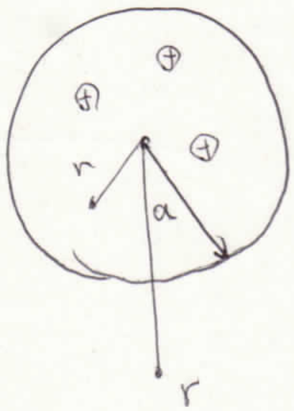
③

ポイント：電荷密度が半径に依存する

↓

一様ではない！



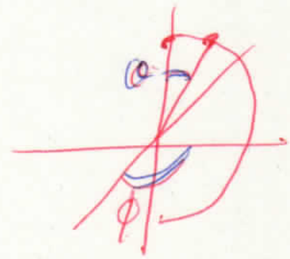
$$\int_S E\cdot ds = \frac{1}{\varepsilon_0}\int_V \rho\, dv$$

i) r<aのとき

$$\rho = \rho_0\left(\frac{r}{a}-1\right)$$

(左辺) $= 4\pi r^2 E =$

(右辺) $= \dfrac{1}{\varepsilon_0}\displaystyle\int_V \rho\, dv = \dfrac{1}{\varepsilon_0}\int_V \rho_0\left(\frac{r}{a}-1\right)dv$



$$= \frac{1}{\varepsilon_0}\int_V \rho_0\left(\frac{r}{a}-1\right) r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$= \frac{\rho_0}{\varepsilon_0}\int_0^r \left(\frac{r}{a}-1\right) r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{r^4}{4a}-\frac{r^3}{3}\right]_0^r \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{r^4}{4a}-\frac{r^3}{3}\right]_0^r \left[1+1\right](2\pi)$$

$$= \frac{4\pi\rho}{\varepsilon_0}\left(\frac{r^4}{4a}-\frac{r^3}{3}\right) \quad \cancel{[V/m]}$$

$$\therefore\ E = \frac{-\cancel{4\pi}\rho}{\cancel{4\pi} r^2 \varepsilon_0}\left(\frac{r^{\cancel{4}\,2}}{4a}-\frac{r^{\cancel{3}}}{3}\right)$$

(ii) $a < r$ のとき (全電荷を内包する)

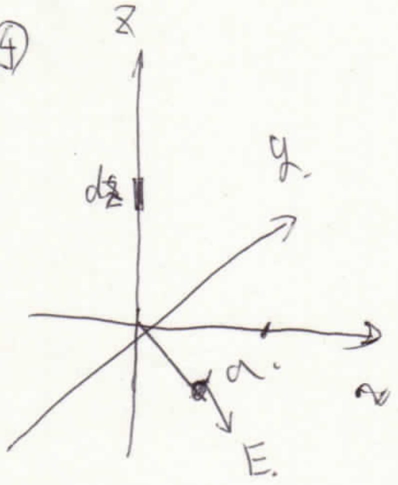
(左辺) $= 4\pi r^2 E$

(右辺) $= -\frac{\rho}{\varepsilon_0}\int_0^a \left(\frac{r}{a} - 1\right) r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$

$$= -\frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^4}{4a} - \frac{r^3}{3}\right]_0^a = -\frac{4\pi\rho_0}{\varepsilon_0}\left(\frac{a^4}{4a} - \frac{a^3}{3}\right) = \frac{\pi\rho_0 a^3}{3\varepsilon_0}$$

$$\therefore E = \frac{\pi\rho_0 a^3}{4\pi r^2 \cdot 3\varepsilon_0} = \frac{\rho_0 a^3}{12\pi\varepsilon_0 r^2}$$

④



(1) $d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2}\frac{\mathbf{r}}{r}$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{(z^2+a^2)^{3/2}}(-z a_z + a a_r)$$

$z a_z + \mathbf{r} = a a_r$

$\mathbf{r} = -z a_z + a a_r$

(2) z 軸全体で積分する

$$\mathbf{E} = \int_{-\infty}^{\infty} d\mathbf{E} = \int_{-\infty}^{\infty}\frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{(z^2+a^2)^{3/2}}(-z a_z + a a_r)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\int_{-\infty}^{\infty}\frac{-z\,dz}{(z^2+a^2)^{3/2}}a_z + a\int_{-\infty}^{\infty}\frac{dz}{(z^2+a^2)^{3/2}}a_r\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}$$

$t^{-\frac{3}{2}+1}$ $t^{?}$

第一項について

$z^2 + a^2 = t$ とおくと $\frac{dt}{dz} = 2z$ $\Leftrightarrow$ $\frac{dt}{2} = z\,dz$

$$-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dt}{t^{\frac{3}{2}}} = -\frac{1}{2}\left[-2t^{-\frac{1}{2}}\right]_{-\infty}^{\infty} = 0$$

第2項について

$z = a\tan\theta$ と おいて

$$\frac{dz}{d\theta} = a\frac{d\theta}{\cos^2\theta}$$

$$\begin{cases} z \to -\infty \\ \theta \to -\frac{\pi}{2} \end{cases} \qquad \begin{cases} z \to +\infty \\ \theta \to \frac{\pi}{2} \end{cases}$$

$$a\int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)^{3/2}} = a\int_{-\pi/2}^{\pi/2} \frac{1}{(a^2\tan^2\theta + a^2)^{3/2}} \cdot a\frac{d\theta}{\cos^2\theta}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{a^2}{(a^2)^{3/2}(\tan^2\theta+1)^{3/2}} \cdot \frac{d\theta}{\cos^2\theta}$$

$$\frac{4}{(2^2)^{3/2}} \qquad \sqrt{4}^3$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{a} \cdot \cos\theta \, d\theta$$

$$\left(\because \frac{1}{(\tan^2\theta+1)^{3/2}} \cdot \frac{1}{\cos^2\theta} = \frac{1}{\left(\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}\right)^{\frac{1}{2}} \cdot \cancel{\left(\frac{1}{\cos^2\theta}\right)}} \cancel{\cos^2\theta} = \frac{\cos\theta}{1} = \cos\theta \right.$$

$$= \frac{1}{a}\Big[+\sin\theta \Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{a}$$

より,

$$E = \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{2}{a} a_r$$

$$= \frac{\lambda}{2\pi\varepsilon_0 a} a_r \ [\mathrm{V/m}]$$